Invariants of 3-manifolds from intersecting kernels of Heegaard splittings

Fengling Li

(Joint with Fengchun Lei and Jie Wu)

School of Mathematical Sciences
Dalian University of Technology, Dalian, China
Email: dutlfl@163.com

RCCKT, July 6, 2017

Content

1. Background
2. Brief review on Heegaard splittings
3. Intersecting Kernel K of a Heegaard Splitting
4. The SCC subgroup $\Lambda(K)$ of K
5. Main results
6. Some questions

1. Background

Let M be a compact connected orientable 3-manifold, $H_1 \cup_S H_2$ a Heegaard splitting of M. For $j = 1, 2$, let

$$i_j : S \hookrightarrow H_j$$

be the inclusion map,

$$i_{j_*} : \pi_1(S) \to \pi_1(H_j)$$

the homomorphism induced by the inclusion, and

$$K_j = \mathrm{Ker}(i_{j_*}).$$

Both K_1 and K_2 are normal subgroups of $\pi_1(S)$. We call $K = K_1 \cap K_2$ the intersecting kernel of the Heegaard splitting $H_1 \cup_S H_2$.

Historical Remarks:

1. The intersecting kernel K was first introduced by J. Stallings in 1960s, as the kernel of the splitting homomorphism, for the reformulation of the Poincaré conjecture in algebraic terms.
2. Stallings' approach has been intensively studied by W. H. Jaco, C. D. Papakyriakopoulos, and E. Rapaport in late 1960s and early 1970s.
3. It was used by J. Birman and others to discover inequivalent Heegaard splittings in 1980s.
4. Some recent work on the subjects of handlebody subgroups in a mapping class group, extending pseudo-Anosov maps into compression bodies, and some others, are also closely related to the intersecting kernels of Heegaard splittings.

Historical Remarks:

1. The intersecting kernel K was first introduced by J. Stallings in 1960s, as the kernel of the splitting homomorphism, for the reformulation of the Poincaré conjecture in algebraic terms.
2. Stallings' approach has been intensively studied by W. H. Jaco, C. D. Papakyriakopoulos, and E. Rapaport in late 1960s and early 1970s.
3. It was used by J. Birman and others to discover inequivalent Heegaard splittings in 1980s.
4. Some recent work on the subjects of handlebody subgroups in a mapping class group, extending pseudo-Anosov maps into compression bodies, and some others, are also closely related to the intersecting kernels of Heegaard splittings.

Historical Remarks:

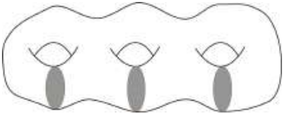
1. The intersecting kernel K was first introduced by J. Stallings in 1960s, as the kernel of the splitting homomorphism, for the reformulation of the Poincaré conjecture in algebraic terms.
2. Stallings' approach has been intensively studied by W. H. Jaco, C. D. Papakyriakopoulos, and E. Rapaport in late 1960s and early 1970s.
3. It was used by J. Birman and others to discover inequivalent Heegaard splittings in 1980s.
4. Some recent work on the subjects of handlebody subgroups in a mapping class group, extending pseudo-Anosov maps into compression bodies, and some others, are also closely related to the intersecting kernels of Heegaard splittings.

Historical Remarks:

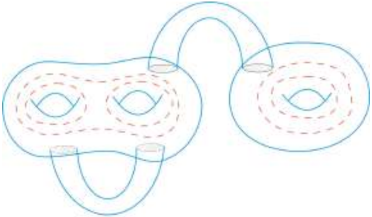
1. The intersecting kernel K was first introduced by J. Stallings in 1960s, as the kernel of the splitting homomorphism, for the reformulation of the Poincaré conjecture in algebraic terms.
2. Stallings' approach has been intensively studied by W. H. Jaco, C. D. Papakyriakopoulos, and E. Rapaport in late 1960s and early 1970s.
3. It was used by J. Birman and others to discover inequivalent Heegaard splittings in 1980s.
4. Some recent work on the subjects of handlebody subgroups in a mapping class group, extending pseudo-Anosov maps into compression bodies, and some others, are also closely related to the intersecting kernels of Heegaard splittings.

2. Brief review on Heegaard splittings

A handlebody is a connected 3-manifold obtained by attaching 1-handles to 3-balls. We may regard a 3-ball as a handlebody of genus 0.



A compression body is a connected 3-manifold obtained by attaching 1-handles to (closed surfaces)$\times I$ and 3-balls.



Compression body

A Heegaard splitting of a compact orientable connected 3-manifold M is a decomposition of M into two compression bodies V and W such that $V \cap W = S = \partial_+ V = \partial_+ W$ and $M = V \cup W$. S is called a Heegaard surface of M.

It is a well-known fact that any compact orientable connected 3-manifold admits a Heegaard splitting.

Heegaard splitting

A Heegaard splitting of a compact orientable connected 3-manifold M is a decomposition of M into two compression bodies V and W such that $V \cap W = S = \partial_+ V = \partial_+ W$ and $M = V \cup W$. S is called a Heegaard surface of M.

It is a well-known fact that any compact orientable connected 3-manifold admits a Heegaard splitting.

Reducible (stabilized) Heegaard splittings

Let $V \cup_S W$ be a Heegaard splitting for M.

$V \cup_S W$ is reducible if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ such that $\partial D_1 = \partial D_2$. Otherwise, $V \cup_S W$ is irreducible.

$V \cup_S W$ is stabilized if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ such that $|\partial D_1 \cap \partial D_2| = 1$. Otherwise, $V \cup_S W$ is unstabilized.

Clearly, a stabilized Heegaard splitting of genus $g \geq 2$ is reducible.

Stabilization of Heegaard splittings

A stabilized Heegaard splitting $V \cup_S W$ can be viewed as a connected sum of a Heegaard splitting $V' \cup_{S'} W'$ (with genus $g(S) - 1$) and a genus 1 Heegaard splitting of S^3. $V \cup_S W$ is called an elementary stabilization of $V' \cup_{S'} W'$.

A Heegaard splitting $V \cup_S W$ is called a stabilization of a Heegaard splitting $V'' \cup_{S''} W''$ if $V \cup_S W$ can be obtained from $V'' \cup_{S''} W''$ by a finite number of elementary stabilizations.

Reidemeister-Singer Theorem

Let $V \cup_S W$ and $V' \cup_{S'} W'$ be two Heegaard splittings for M.

$V \cup_S W$ and $V' \cup_{S'} W'$ are called equivalent if S and S' are isotopic in M.

$V \cup_S W$ and $V' \cup_{S'} W'$ are called stably equivalent if, after a finite number of elementary stabilizations, they have a common stabilization up to equivalence.

Theorem (Reidemeister-Singer Theorem)

Any two Heegaard splittings $V \cup_S W$ and $V' \cup_{S'} W'$ for 3-manifold M are stably equivalent.

Definition

Let M be a 3-manifold and $\mathcal{M} = (M; H_1, H_2; S)$ a Heegaard splitting for M. Let $i_j : S \hookrightarrow H_j$ be the inclusion, and $i_{j*} : \pi_1(S) \to \pi_1(H_j)$ the induced homomorphism, $j = 1, 2$. Then $\text{Ker}(i_{1*}) \cap \text{Ker}(i_{2*})$ is called the *intersecting kernel* of $\mathcal{M}$, and is denoted by $K(\mathcal{M})$.

Clearly $K(\mathcal{M})$ is a (normal) subgroup of $\pi_1(S)$, which is a *Fuchsian*-group. It is a well-known fact that every subgroup of $\pi_1(S)$ with finite index (infinite index, resp.) is a *Fuchsian*-group (free group, resp.).

3. Intersecting Kernels of Heegaard Splittings

Definition

Let M be a 3-manifold and $\mathcal{M} = (M; H_1, H_2; S)$ a Heegaard splitting for M. Let $i_j : S \hookrightarrow H_j$ be the inclusion, and $i_{j*} : \pi_1(S) \to \pi_1(H_j)$ the induced homomorphism, $j = 1, 2$. Then $\mathrm{Ker}(i_{1*}) \cap \mathrm{Ker}(i_{2*})$ is called the *intersecting kernel* of $\mathcal{M}$, and is denoted by $K(\mathcal{M})$.

Clearly $K(\mathcal{M})$ is a (normal) subgroup of $\pi_1(S)$, which is a *Fuchsian*-group. It is a well-known fact that every subgroup of $\pi_1(S)$ with finite index (infinite index, resp.) is a *Fuchsian*-group (free group, resp.).

Simple examples

1. Let $\mathcal{M} = (S^3; H_1, H_2; T)$ be a genus 1 Heegaard splitting for S^3. Let a, b be two essential simple closed curves on the torus T such that a bounds a disk in H_1, b bounds a disk in H_2, and a and b intersect in a single point P, which we choose as a base point. Then $\{[a], [b]\}$ is a basis for $\pi_1(T)$. Clearly,

$$\mathrm{Ker}(i_{1*} : \pi_1(T) \to \pi_1(H_1)) = \{n[a] : n \in \mathbb{Z}\},$$

$$\mathrm{Ker}(i_{2*} : \pi_1(T) \to \pi_1(H_2)) = \{n[b] : n \in \mathbb{Z}\}.$$

   Thus $K(\mathcal{M}) = \{0\}$. Similarly,

2. for a genus 1 Heegaard splitting $\mathcal{M}_1$ for a lens space $L(p, q)$, we have $K(\mathcal{M}_1) = \{0\}$;
3. for a genus 1 Heegaard splitting $\mathcal{M}_2$ for $S^2 \times S^1$, we have $K(\mathcal{M}_2) \cong \mathbb{Z}$.

Some properties

Proposition

Let $V \cup_S W$ be a non-trivial Heegaard splitting of genus ≥ 2 for M. Let $i : S \hookrightarrow V$ and $j : S \hookrightarrow W$ be the inclusions, and $i_* : \pi_1(S) \to \pi_1(V)$, $j_* : \pi_1(S) \to \pi_1(W)$ the induced homomorphisms. Then for any $\alpha \in \mathrm{Ker} i_*$, $\beta \in \mathrm{Ker} j_*$, $[\alpha, \beta] \in K(V \cup_S W)$. In other words, $[\mathrm{Ker} i_*, \mathrm{Ker} j_*] \lhd K(V \cup_S W)$.

Remark. For a non-trivial Heegaard splitting $\mathcal{M}$ of genus ≥ 2, $K(\mathcal{M})$ is never trivial.

Proposition

A Heegaard splitting $\mathcal{M} = (M; V, W; S)$ is reducible if and only if there exists an essential simple closed curve C in S such that $[C] \in K(\mathcal{M})$.

Proposition

Let $V \cup_S W$ be a non-trivial Heegaard splitting of genus ≥ 2 for M. Let $i : S \hookrightarrow V$ and $j : S \hookrightarrow W$ be the inclusions, and $i_* : \pi_1(S) \to \pi_1(V)$, $j_* : \pi_1(S) \to \pi_1(W)$ the induced homomorphisms. Then for any $\alpha \in \mathrm{Ker} i_*$, $\beta \in \mathrm{Ker} j_*$, $[\alpha, \beta] \in K(V \cup_S W)$. In other words, $[\mathrm{Ker} i_*, \mathrm{Ker} j_*] \lhd K(V \cup_S W)$.

Remark. For a non-trivial Heegaard splitting $\mathcal{M}$ of genus ≥ 2, $K(\mathcal{M})$ is never trivial.

Proposition

A Heegaard splitting $\mathcal{M} = (M; V, W; S)$ is reducible if and only if there exists an essential simple closed curve C in S such that $[C] \in K(\mathcal{M})$.

Some properties

$K(\mathcal{M})$ contains important topological information as follows:

Theorem (Lei-Wu, 2012)

Let $H \cup_S H'$ be a Heegaard splitting for a closed orientable 3-manifold M. Let $i : S \hookrightarrow H$, $i' : S \hookrightarrow H'$, $i_* : \pi_1(S) \to \pi_1(H)$ and $i'_* : \pi_1(S) \to \pi_1(H')$ be as before. Then subject to the positive solution to Poincaré conjecture, we have

$$\frac{\mathrm{Ker}\, i_* \cap \mathrm{Ker}\, i'_*}{[\mathrm{Ker}\, i_*, \mathrm{Ker}\, i'_*]} \cong \pi_2(M).$$

Some Properties

Theorem (Lei-Wu, 2012)

Let $\mathcal{M}_1 = (M_1; V_1, W_1; S_1)$, $\mathcal{M}_2 = (M_2; V_2, W_2; S_2)$ be two Heegaard splittings, and $\mathcal{M} = \mathcal{M}_1 \#_{S^2} \mathcal{M}_2 = (M; V, W; S)$. Then there is a short exact sequence of groups

$$1 \to \langle [C] \rangle^N \to K(\mathcal{M}) \to K(\mathcal{M}_1) * K(\mathcal{M}_2) \to 1,$$

where C is the intersecting curve of the 2-sphere S^2 and the Heegaard surface S.

A Corollary

Applying above theorem to a stabilized Heegaard splitting, we have

Corollary (Lei-Wu, 2012)

Let $\mathcal{M}' = (M; V', W'; S')$ be an elementary stabilization of the Heegaard splitting $\mathcal{M} = (M; V, W; S)$. Then there is a short exact sequence of groups

$$1 \to \langle [C] \rangle^N \to K(\mathcal{M}') \to K(\mathcal{M}) \to 1,$$

where C is the intersecting curve of S' with the 2-sphere S^2, which realizes the connected sum decomposition.

In particular, for the genus 2 splitting $\mathcal{M}' = (S^3; V, W; S)$ for S^3, we have $K(\mathcal{M}') \cong \langle [C] \rangle^N$, where C is a s.c.c. on S, s.t. C cuts S into two once punctured tori and C bounds disks in both V and W.

A Corollary

Applying above theorem to a stabilized Heegaard splitting, we have

Corollary (Lei-Wu, 2012)

Let $\mathcal{M}' = (M; V', W'; S')$ be an elementary stabilization of the Heegaard splitting $\mathcal{M} = (M; V, W; S)$. Then there is a short exact sequence of groups

$$1 \to \langle [C] \rangle^N \to K(\mathcal{M}') \to K(\mathcal{M}) \to 1,$$

where C is the intersecting curve of S' with the 2-sphere S^2, which realizes the connected sum decomposition.

In particular, for the genus 2 splitting $\mathcal{M}' = (S^3; V, W; S)$ for S^3, we have $K(\mathcal{M}') \cong \langle [C] \rangle^N$, where C is a s.c.c. on S, s.t. C cuts S into two once punctured tori and C bounds disks in both V and W.

4. The SCC subgroup $\Lambda(K)$ of K

Let $\mathcal{M} = (M; V, W; S)$ be a Heegaard splitting for a 3-manifold M, and K the intersecting kernel.

Consider the normal subgroup

$$\Lambda(K) = \langle [\alpha] \in K : \alpha \text{ is an essential simple closed curve on } S \rangle^N$$

of K. We call $\Lambda(K)$ the SCC subgroup of K.

As we see before, a Heegaard splitting $\mathcal{M}$ is reducible if and only if $\Lambda(K)$ is non-trivial.

Let $\mathcal{M}' = (M; V', W'; S')$ be an elementary stabilization of Heegaard splitting $\mathcal{M} = (M; V, W; S)$ for M. By the previous corollary, there is a surjective homomorphim $h : K(\mathcal{M}') \twoheadrightarrow K(\mathcal{M})$.

Note that $\Lambda(K(\mathcal{M})) \subset \Lambda(K(\mathcal{M}'))$, there exists a commutative diagram

$$(*) \qquad \begin{array}{ccc} K(\mathcal{M}') & \xrightarrow{\;h\;} & K(\mathcal{M}) \\ \downarrow q' & & \downarrow q \\ K(\mathcal{M}')/\Lambda(K(\mathcal{M}')) & \xleftarrow{\;\rho\;} & K(\mathcal{M})/\Lambda(K(\mathcal{M})) \end{array}$$

Quotient group $QK(\mathcal{M})$

In general, set

$\mathcal{M}^{(0)} = \mathcal{M}$, $\mathcal{M}^{(1)} = \mathcal{M}'$, $\cdots\cdots$, $\mathcal{M}^{(n)} = (\mathcal{M}^{(n-1)})'$, $n \in \mathbb{N}$,

and $\rho_i : K(\mathcal{M}^{(i)})/\Lambda(K(\mathcal{M}^{(i)})) \twoheadrightarrow K(\mathcal{M}^{(i+1)})/\Lambda(K(\mathcal{M}^{(i+1)}))$, $i = 0, 1, 2, \cdots$.

We have a sequence of surjective homomorphisms

$$K(\mathcal{M}^{(0)})/\Lambda(K(\mathcal{M}^{(0)})) \twoheadrightarrow K(\mathcal{M}^{(1)})/\Lambda(K(\mathcal{M}^{(1)})) \twoheadrightarrow \cdots$$
$$\cdots \twoheadrightarrow K(\mathcal{M}^{(n)})/\Lambda(K(\mathcal{M}^{(n)})) \twoheadrightarrow \cdots$$

The direct limit $\varinjlim_{n\in\mathbb{N}} K(\mathcal{M}^{(n)})/\Lambda(K(\mathcal{M}^{(n)}))$ is a (in general, non-trivial) group, which is denoted by $QK(\mathcal{M})$.

5. Main Results

Theorem (L-Lei-Wu, 2016)

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be any two Heegaard splittings of a closed orientable 3-manifold M. Then

$$QK(\mathcal{M}_1) \cong QK(\mathcal{M}_2).$$

Remark: By the above theorem, for any Heegaard splitting $\mathcal{M}$ of a 3-manifold M, $QK(\mathcal{M})$ is independent of the choice of the Heegaard splitting, therefore it defines an invariant of M.

Theorem (L-Lei-Wu, 2016)

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be any two Heegaard splittings of a closed orientable 3-manifold M. Then

$$QK(\mathcal{M}_1) \cong QK(\mathcal{M}_2).$$

Remark: By the above theorem, for any Heegaard splitting $\mathcal{M}$ of a 3-manifold M, $QK(\mathcal{M})$ is independent of the choice of the Heegaard splitting, therefore it defines an invariant of M.

Some Corollaries

Corollary

For any compact orientable 3-manifold M, $QK(M)$ is an invariant of the 3-manifold.

This corollary has an interesting application in knot theory.

Let K be a knot in S^3, and $E(K)$ the knot exterior of K.

From the above corollary, $QK(E(K))$ is an invariant of the 3-manifold $E(K)$. Since knots are determined by their complements, $QK(E(K))$ is an invariant of the knot K.

We get a knot invariant from the Heegaard splitting of its complement.

Corollary

For any compact orientable 3-manifold M, $QK(M)$ is an invariant of the 3-manifold.

This corollary has an interesting application in knot theory.

Let K be a knot in S^3, and $E(K)$ the knot exterior of K.

From the above corollary, $QK(E(K))$ is an invariant of the 3-manifold $E(K)$. Since knots are determined by their complements, $QK(E(K))$ is an invariant of the knot K.

We get a knot invariant from the Heegaard splitting of its complement.

We may further have

Corollary

For any compact orientable 3-manifold M, $QK(M)^{ab}$ is an invariant of the 3-manifold.

If $QK(M)$ is trivial for some 3-manifold M, the SCC subgroup can be used to detect the intersecting kernel.

Corollary

Let M be a compact orientable 3-manifold. If $QK(M) = \{1\}$, then there exists a Heegaard splitting $\mathcal{M}$ of M such that the intersecting kernel of $\mathcal{M}$ is isomorphic to its SCC subgroup.

Example

For $M = S^3$, $QK(M) = 1$.

(1) For genus 0 splitting $\mathcal{M}_0$ of S^3, it is obvious that $K(\mathcal{M}_0)$ is trivial, then $\Lambda(K(\mathcal{M}_0))$ is trivial.

(2) For genus 1 splitting $\mathcal{M}_1$ of S^3, $K(\mathcal{M}_1)$ is trivial, then $\Lambda(K(\mathcal{M}_1))$ is also trivial.

(3) For genus 2 splitting $\mathcal{M}_2$ of S^3, $K(\mathcal{M}_2) = \langle[C]\rangle^N$ $(\cong \mathbb{Z})$, while $K(\mathcal{M}_2)/\Lambda(K(\mathcal{M}_2))$ is trivial, hence $\Lambda(K(\mathcal{M}_2)) \cong K(\mathcal{M}_2) \cong \mathbb{Z}$.

Some examples

Example

For $M = S^3$, $QK(M) = 1$.

(1) For genus 0 splitting $\mathcal{M}_0$ of S^3, it is obvious that $K(\mathcal{M}_0)$ is trivial, then $\Lambda(K(\mathcal{M}_0))$ is trivial.

(2) For genus 1 splitting $\mathcal{M}_1$ of S^3, $K(\mathcal{M}_1)$ is trivial, then $\Lambda(K(\mathcal{M}_1))$ is also trivial.

(3) For genus 2 splitting $\mathcal{M}_2$ of S^3, $K(\mathcal{M}_2) = \langle [C] \rangle^N$ ($\cong \mathbb{Z}$), while $K(\mathcal{M}_2)/\Lambda(K(\mathcal{M}_2))$ is trivial, hence $\Lambda(K(\mathcal{M}_2)) \cong K(\mathcal{M}_2) \cong \mathbb{Z}$.

Example

For $M = S^2 \times S^1$.
Let $\mathcal{M}_1 = (M; H_1, H_1'; T)$ be the genus 1 splitting for M. As we see before, $K(\mathcal{M}_1) \cong \mathbb{Z}$, it is generated by $[\alpha]$, where α is a s.c.c. which bounds a meridian disk in each solid torus. Since $K(\mathcal{M}_1) \lhd \pi_1(T)$, $K(\mathcal{M}_1) = \langle [\alpha] \rangle = \langle [\alpha] \rangle^N$. α is essential in T, so $[\alpha] \in K(\mathcal{M}_1)$, which implies that $\langle [\alpha] \rangle^N \lhd \Lambda(K(\mathcal{M}_1))$. But $\Lambda(K(\mathcal{M}_1)) \lhd K(\mathcal{M}_1)$, so $\Lambda(K(\mathcal{M}_1)) \cong \langle [\alpha] \rangle^N$, and hence $\Lambda(K(\mathcal{M}_1)) \cong K(\mathcal{M}_1)$. Thus we have

$$K(\mathcal{M}_1)/\Lambda(K(\mathcal{M}_1)) = 1.$$

So $QK(S^2 \times S^1)$ is trivial.

Some Questions

1. Classify the 3-manifolds M with $QK(M) = 1$.

2. Give examples of the 3-manifolds M with $QK(M) \neq 1$.

3. Determine the algebraic structures of the group $QK(M)$ for a 3-manifold M. Is $QK(M)$ residually nilpotent? If so, what is the Lie algebra of $QK(M)$?

4. Are there any relations between $QK(M)$ and the other known invariants of M?

THANKS FOR YOUR ATTENTION!